Antipercolation on Bethe and triangular lattices

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Abstract. Antipercolation is related to percolation somewhat as antiferromagnets are to ferromagnets.

The antipercolation problem is solved on the Bethe lattice. The critical exponents are identical to the percolation exponents when the coordination number z is greater than a critical value $z_c = 3$, for which the problem has new exponents satisfying extended universality and below which there is no transition. For alternate lattices the problem may be transformed into a percolation problem with different occupation probabilities on the two sublattices. This allows a connection with an s-state Potts model with different z-spin interactions on the two sublattices.

In two dimensions there is no transition on the alternate square and honeycomb lattices whereas a transition exists on the triangular lattice. Using the phenomenological renormalisation group method, the critical concentration is found to be $p_c^{(a)} = 0.21$ whereas the correlation length exponent ν_a seems to converge towards the accepted percolation value $\nu_p = 1.333$.

1. Introduction

In the site percolation problem each site on a lattice is either black or white with independent probabilities p and $q = 1 - p$. The quantities of interest are the critical probability $p_c^{(p)}$ for which an infinite cluster first appears, the mean number of clusters per site $F^{(p)}(p)$, the percolation probability $P^{(p)}(p)$ and the mean square finite cluster size $S^{(p)}(p)$ (Shante and Kirkpatrick 1971, Essam 1972, 1980, Stauffer 1979).

In this work we present some results concerning the antipercolation (AP) problem. Its relation to the percolation problem is similar to that between antiferromagnetism and ferromagnetism. In the AP problem, two neighbouring sites are directly connected and belong to the same bond cluster when they have different colours so that AP bond clusters contain alternating black and white sites (see figure 1(a)). A site surrounded by z first neighbours of the same colour (z is the coordination number) builds up a one-site cluster. The problem is symmetrical under an interchange of the colours ($p \leftrightarrow q$) so that when an infinite cluster appears at the lower critical probability $p_c^{(a)} \leq \frac{1}{2}$ the AP probability $P^{(a)}(p)$ is non-zero until p reaches the upper critical probability $q_c^{(a)} = 1 - p_c^{(a)}$. The functions $F^{(a)}(p)$, $P^{(a)}(p)$ and $S^{(a)}(p)$ retain the same meaning as above. They are symmetrical about $p = \frac{1}{2}$:

$$X^{(a)}(p) = X^{(a)}(q) \qquad (X = F, P, S). \tag{1.1}$$

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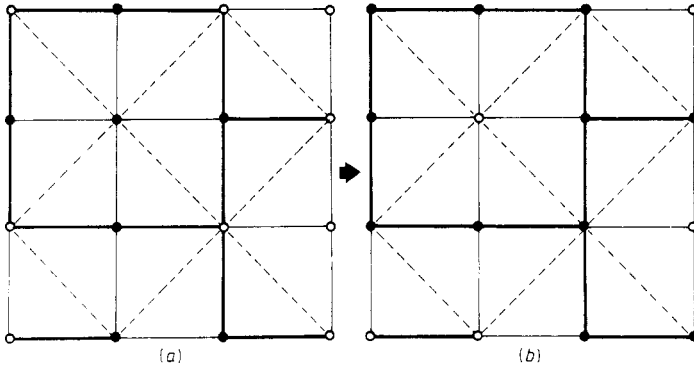‡ Laboratoire associé au CNRS no 155.

Figure 1. (*a*) Antipercolation on an alternate lattice: clusters (bold lines) are built up of neighbouring sites with different colours. (*b*) When the colour of the sites belonging to sublattice (A) (broken lines) is changed, then either a black or a white percolation cluster corresponds to each antipercolation cluster on the initial lattice.

The AP model may describe either the behaviour at zero temperature and zero field of a hypothetical magnetic alloy A_pB_{1-p} in which the exchange interactions $J_{AA} = J_{BB} = 0$ whereas $J_{AB} \neq 0$ or gelation in a solution with two types A and B of molecules where only A–B bonds are possible.

2. Antipercolation on the Bethe lattice

The AP problem is easily solved on the Bethe lattice using the ghost field and branch function method (Marland and Stinchcombe 1977, Reynolds *et al* 1977, Stinchcombe 1974, Turban 1979). Details are given in the appendix. The critical probability is given by:

$$p_c^{(a)} = \tfrac{1}{2} - (\tfrac{1}{4} - 1/(z-1)^2)^{1/2} \tag{2.1}$$

so that a transition to the antipercolating phase occurs only when the coordination number z exceeds a critical value $z_c = 3$ for which $p_c^{(a)} = q_c^{(a)} = \tfrac{1}{2}$. The critical probability $p_c^{(a)}$ may be rewritten as a function of the site percolation threshold since $p_c^{(p)} = 1/(z-1)$ on the Bethe lattice:

$$p_c^{(a)} = \tfrac{1}{2} - (\tfrac{1}{4} - p_c^{(p)2})^{1/2} \tag{2.2}$$

giving a better approximation for alternate lattices in lower dimensionalities on which there is no transition when $p_c^{(p)} > \tfrac{1}{2}$ (see § 3).

Near the threshold the AP probability is given by

$$P^{(a)}(p) = \frac{2z(z-1)(1-2p_c^{(a)})}{z-2}\varepsilon + O(\varepsilon^2) \qquad (z>3) \tag{2.3}$$

$$P^{(a)}(p) = 0 \qquad (\text{all } p, z = 3) \tag{2.4}$$

where $\varepsilon = p - p_c^{(a)}$. The mean square cluster size is

$$S^{(a)}(p) = \frac{z(z+1)}{(z-1)^4(1-2p_c^{(a)})}\varepsilon^{-1} + O(1) \qquad (z>3) \tag{2.5}$$

$$S^{(a)}(p) = \tfrac{3}{4}\varepsilon^{-2} + O(1) \qquad (z=3) \tag{2.6}$$

and at the AP threshold

$$P^{(a)}(p_c^{(a)}, h) = z\left(\frac{2p_c^{(a)}}{z-2}\right)^{1/2}\left(\frac{z}{z-1} - p_c^{(a)}\right)h^{1/2} + O(h) \qquad (z \geq 3) \tag{2.7}$$

where h is the ghost field (see the appendix).

It follows that when $z > z_c$, the critical exponents

$$\beta_a = 1 \qquad \gamma_a = 1 \qquad \delta_a = 2 \tag{2.8}$$

are identical to the percolation exponents.

When $z = z_c$, we find

$$\gamma_a' = 2 \qquad \delta_a' = 2 \tag{2.9}$$

and, using the Griffiths scaling law $\gamma = \beta(\delta - 1)$,

$$\beta_a' = 2 \tag{2.10}$$

in agreement with the extended universality (Suzuki 1974). The change of exponents when $p_c^{(a)} = \frac{1}{2}$ is linked to the (p, q) symmetry of the problem (see § 3 and Burchard 1972).

3. Alternate lattices and the Potts model

When the lattice is built up of two interpenetrating identical sublattices (A) and (B) (figure 1), by changing the colours on one of the sublattices, say (A), we recover a site percolation problem, since to any alternate cluster on the starting lattice (figure 1(a)) there corresponds either a white or a black percolation cluster on the transformed lattice (figure 1(b)). If p is the black site occupation probability $p(b)$ and q the white site occupation probability $p(w)$ on the starting lattice, then on the transformed lattice the probabilities remain unchanged on the (B) sublattice whereas $p(b) = q$ and $p(w) = p$ on the (A) sublattice. The concentrations of black and white sites are the same on the transformed lattice;

$$c(b) = c(w) = \frac{1}{2}(p + q) = \frac{1}{2}. \tag{3.1}$$

It follows that black and white clusters have the same properties.

When $p = \frac{1}{2}$ we recover the usual site percolation problem and

$$S^{(a)}(\tfrac{1}{2}) = S^{(p)}(\tfrac{1}{2}) \tag{3.2}$$

$$P^{(a)}(\tfrac{1}{2}) = 2P^{(p)}(\tfrac{1}{2}) \tag{3.3}$$

where $P^{(p)}(p)$ is the probability for any site to be black and to belong to the infinite black cluster (the factor of two is due to the possibility of black or white infinite clusters on the transformed lattice). Alternate lattices with $P^{(p)}(\tfrac{1}{2}) = 0$ do not present any AP transition. In two dimensions this occurs for the honeycomb and square lattices. The triangular lattice for which $p_c^{(p)} = \frac{1}{2}$ is not alternate and the argument given above no longer holds, but it does present a transition as will be shown in the following section. In three dimensions the simple cubic and diamond lattices have an AP transition and other lattices with higher coordinate numbers are expected to have one as well.

To get the AP critical exponents on alternate lattices one may use the connection mentioned above between AP and site percolation and study the statistics of black

clusters on the transformed lattice since black and white clusters have the same properties. The site percolation problem on a lattice G with coordination number z has been shown to be related to the $s = 1$ limit of an s-state Potts model with z-spin interactions on the covering lattice G_c (Giri *et al* 1977, Kunz and Wu 1978). On the transformed lattice we have a percolation problem with black site probability q on the (A) sublattice and p on the (B) sublattice. It follows that the corresponding Potts model has different coupling constants K_1 and K_2 ($K = J/k_B T$) on the two sublattices with (see figure 2)

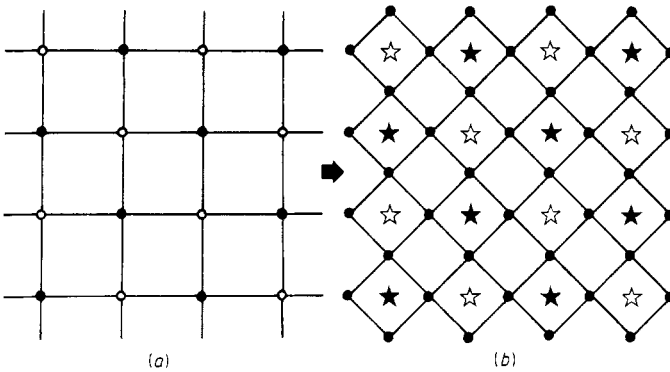$$q = 1 - \exp(-K_1) \tag{3.4}$$

$$p = 1 - \exp(-K_2). \tag{3.5}$$

Figure 2. (a) The square lattice G is divided into two square sublattices (with white or black sites). (b) On the covering lattice G_c we have four-spin interactions K_1 (white stars) or K_2 (black stars) for the corresponding Potts model.

The physical line in the (K_1, K_2) plane is such that

$$\exp(-K_1) + \exp(-K_2) = 1. \tag{3.6}$$

Let $f(K_{1c}, K_{2c}) = 0$ be the critical line of the Potts model in the $s = 1$ limit; the physical and critical lines are both symmetrical in the K_1 and K_2 variables and may have, depending on the lattice, either no intersection (i.e. no AP transition) (figure 3(a)),
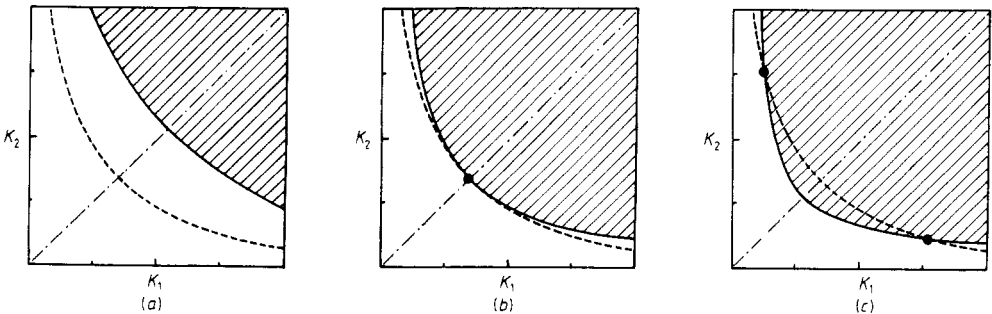

Figure 3. In the (K_1, K_2) plane, the critical line (full curve) limits the percolating phase (hatched area). The physical line (broken line) may have (a) no intersection with the critical line, (b) one intersection corresponding to $p_c^{(a)} = \frac{1}{2}$, (c) two symmetrical intersections corresponding to the lower and upper critical probabilities for antipercolation.

one intersection on the line $K_1 = K_2$ corresponding to $p_c^{(a)} = \frac{1}{2}$ (figure 3(b)) or two symmetrical intersections $p_c^{(a)}$ and $q_c^{(a)}$ (figure 3(c)).

In the last case, if the perturbation $K = \frac{1}{2}(K_1 - K_2)$ on the site percolation problem is irrelevant, the critical behaviour will be governed by the percolation fixed point, whereas when the two critical points coalesce at $p_c^{(a)} = \frac{1}{2}$ (figure 3(b)) the critical point is approached tangentially along the physical line and the temperature-like scaling field is of the second order in $(p - p_c^{(a)})$, leading to extended universality.

4. Antipercolation on the triangular lattice—phenomenological renormalisation group approach

As well as a transition for the site percolation problem with $p_c^{(p)} = \frac{1}{2}$, the triangular lattice also shows an AP transition. Namely, all the boundary sites of a site percolation cluster belong to the same AP cluster. When $p = p_c^{(p)}$ the size of a percolation cluster diverges, which is also true for its boundary. Thus an infinite AP cluster exists at $p = \frac{1}{2}$. However, as may be verified from the computer simulations shown in figure 4, the AP transition on the triangular lattice occurs at a much lower concentration.
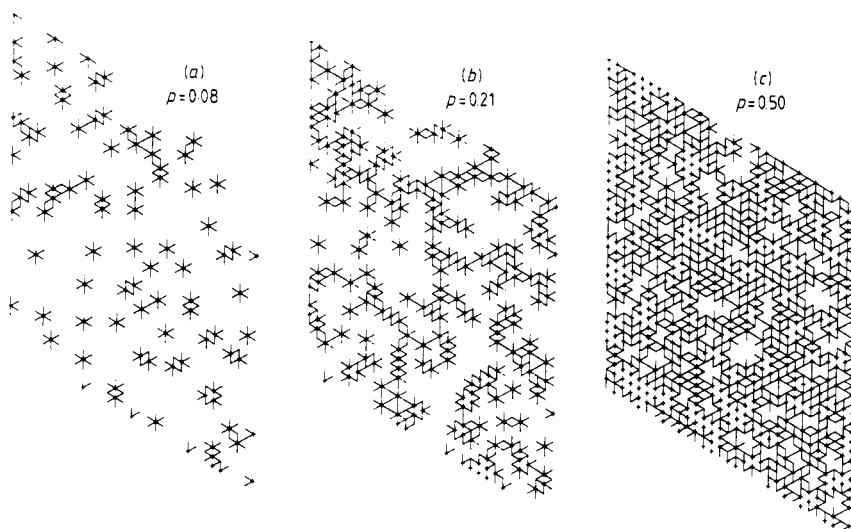

Figure 4. Computer simulation of the antipercolation problem on the triangular lattice with $p = 0.08$ (a), $p = 0.21$ (b) and $p = 0.50$ (c). +, occupied site.

The critical concentration $p_c^{(a)}$ and the correlation length exponent ν_a for the AP problem on the triangular lattice have been obtained using the phenomenological renormalisation group approach (Nightingale 1977, 1979, Sneddon 1978, Nightingale and Blöte 1980, Derrida and Vannimenus 1980, Derrida and de Seze 1982).

The two-dimensional lattice is replaced by an infinite strip, n sites wide, and periodic boundary conditions are used. The strip correlation length ξ_n is calculated using the transfer matrix technique. All distinct configurations making use of the circular permutation symmetry of a column of n sites are retained. For instance, a
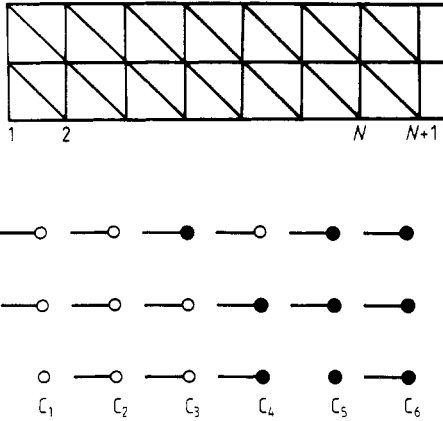
Figure 5. Lattice strip used in the transfer matrix calculation. With $n = 3$ and periodic boundary conditions we get six distinct configurations (C_1 to C_6) when the permutation symmetry is used. Black or white sites with a line are connected to the origin.

column of three sites with periodic boundary conditions gives the six possible configurations shown in figure 5. The configurations which are not connected to the first column are deleted since we want to get the correlation function $\Gamma(N)$ i.e. the probability of having any site on column N connected to the first column. The transfer matrix elements T_{ij} are then determined as the probabilities of getting the configuration C_j on the $(M + 1)$th column given the configuration C_i on the Mth column (Derrida and Vannimenuš 1980, Derrida and de Seze 1982), and with $n = 3$ we get the transfer matrix

$$T_3 = \begin{vmatrix} 0 & 0 & 3pq^2 & 3p^2q & 0 & p^3 \\ 0 & 0 & 3pq^2 & 3p^2q & 0 & p^3 \\ q^3 & 0 & 3pq^2 & 3p^2q & 0 & p^3 \\ 0 & q^3 & 3pq^2 & 3p^2q & p^3 & 0 \\ 0 & q^3 & 3pq^2 & 3p^2q & 0 & 0 \\ 0 & q^3 & 3pq^2 & 3p^2q & 0 & 0 \end{vmatrix}. \tag{4.1}$$

The largest eigenvalue $\lambda_{n\,\max}$ of the transfer matrix is related to the correlation length ξ_n since

$$\Gamma_n(N) \sim \lambda_{n\,\max}^N \sim \exp(-N/\xi_n) \tag{4.2}$$

so that

$$\xi_n = -1/\ln(\lambda_{n\,\max}). \tag{4.3}$$

The phenomenological renormalisation group equation relates the occupation probability p' on a strip of width n to its value p on a strip of width $m > n$:

$$\xi_n(p')/n = \xi_m(p)/m. \tag{4.4}$$

The critical concentration $p_c^{(a)}$ is a fixed point of this equation for both n and m tending to infinity:

$$\xi_n(p_c^{(a)})/n = \xi_m(p_c^{(a)})/m \qquad n, m \to \infty. \tag{4.5}$$

Table 1. Approximate critical concentrations $p_c^{(a)}(n, m)$ and correlation length exponents $\nu_a(n, m)$ for the antipercolation problem on the triangular lattice. S_m gives the size of the transfer matrix for the strip of width m. The corresponding site percolation results of Derrida and de Seze (1982) are given for comparison.

$n-m$	S_m	$p_c^{(a)}(n, m)$	$\nu_a(n, m)$	$p_c^{(p)}(n, m)$	$\nu_p(n, m)$
1–2	2	0.141 83	0.8494	0.733 89	1.5604
2–3	6	0.149 68	1.4717	0.582 14	1.4932
3–4	16	0.210 90	1.4167	0.590 96	1.4718
4–5	38	0.209 18	1.4168	0.588 70	1.4139
5–6	105	0.212 07	1.3808	0.589 92	1.3880

Since our calculations are limited to narrow strips by the computer time available, the critical concentrations $p_c^{(a)}(n, m)$ obtained are subject to finite size errors. These can be minimised by taking $m = n + 1$ (Derrida and de Seze 1982). As seen from table 1 the resulting values for the AP threshold do not show a monotonic convergence, but already a value $p_c^{(a)} \simeq 0.21$ may be estimated from the strips of width up to $n = 6$.

The correlation length exponent ν_a may be deduced from the relation (Derrida and de Seze 1982)

$$1 + \frac{1}{\nu_a} = \frac{\ln[(\partial \xi_n / \partial p)_{p_c^{(a)}} / (\partial \xi_m / \partial p)_{p_c^{(a)}}]}{\ln(n/m)} \tag{4.6}$$

and is also given in table 1. Larger sizes would be needed to extrapolate to infinite width but, in any case, the calculated values indicate that the critical exponent ν_a decreases with increasing strip width towards its percolation value $\nu_p = 1.333$ (Den Nijs 1979). These values for narrow strips show the same evolution with strip width as those of Derrida and de Seze (1982) for the site percolation problem with the same boundary conditions (table 1). This suggests that both problems belong to the same universality class.

Appendix. Ghost field and branch function method on the Bethe lattice

On a lattice with N sites which are black (white) with probability p $(1-p)$, let N_n be the number of bond clusters of n sites, two neighbouring sites being directly connected when their colours are different. The probability $\mathcal{P}_n(p)$ for a given site to belong to an n-site cluster is

$$\mathcal{P}_n(p) = \lim_{N \to \infty} nN_n/N. \tag{A1}$$

The AP probability $P^{(a)}(p)$ is given by

$$P^{(a)}(p) = 1 - \sum_n{}' \mathcal{P}_n(p) \tag{A2}$$

where the prime on the sum indicates the exclusion of the infinite cluster. Equation (A2) simply states that any site belongs either to the infinite cluster with probability $P^{(a)}(p)$ or to any one of the finite clusters.

The mean square cluster size $S^{(a)}(p)$ is given by

$$S^{(a)}(p) = \lim_{N \to \infty} {\sum_n}' n^2 N_n/N = {\sum_n}' n \mathscr{P}_n(p). \tag{A3}$$

Let us now introduce a ghost site which is connected to each of the lattice sites with probability h. Then

$$\mathscr{P}_n(p, h) = \mathscr{P}_n(p)(1-h)^n \tag{A4}$$

since a cluster remains an n-site cluster only when all its sites are disconnected from the ghost site and this occurs with probability $(1-h)^n$. Then

$$P^{(a)}(p, h) = 1 - {\sum_n}' \mathscr{P}_n(p, h) \tag{A5}$$

and $S^{(a)}(p)$ may be rewritten as

$$S^{(a)}(p) = \frac{\partial}{\partial h} P^{(a)}(p, h) \Big|_{h=0}. \tag{A6}$$

In the AP problem $P^{(a)}(p)$ is the order parameter, $S^{(a)}(p)$ the susceptibility and h, which allows us to count the black and white sites, is similar to a staggered field.

Let $R_1(p, h)$ $(R_2(p, h))$ be the probability that a branch, on the Bethe lattice with coordination number z, starting on a black (white) site is finite; clearly

$$R_1(p, h) = p + (1-p)(1-h)(R_2(p, h))^{z-1} \tag{A7}$$

$$R_2(p, h) = 1 - p + p(1-h)(R_1(p, h))^{z-1} \tag{A8}$$

and

$$1 - P^{(a)}(p, h) = (1-h)[p(R_1(p, h))^z + (1-p)(R_2(p, h))^z]. \tag{A9}$$

Near the AP threshold, we may write

$$R_{1(2)}(p) = 1 - \eta_{1(2)} \tag{A10}$$

and up to terms of the first order in η_1, η_2 and $\varepsilon = p - p_c^{(a)}$

$$\begin{aligned} \eta_1 - (z-1)(1 - p_c^{(a)})\eta_2 &= 0 \\ p_c^{(a)}(z-1)\eta_1 - \eta_2 &= 0 \end{aligned} \tag{A11}$$

so that

$$p_c^{(a)} = \tfrac{1}{2} - (\tfrac{1}{4} - 1/(z-1)^2)^{1/2} \qquad q_c^{(a)} = 1 - p_c^{(a)}. \tag{A12}$$

Keeping terms of second order leads to

$$\eta_1 = \frac{2(1 - 2p_c^{(a)})\varepsilon}{(z-2)p_c^{(a)}(1 - p_c^{(a)})[1 + (z-1)p_c^{(a)}]} + O(\varepsilon^2)$$

$$\eta_2 = p_c^{(a)}(z-1)\eta_1 + O(\varepsilon^2) \tag{A13}$$

so that when $z > 3$

$$P^{(a)}(p) = \frac{2z(z-1)(1 - 2p_c^{(a)})}{z-2}\varepsilon + O(\varepsilon^2) \tag{A14}$$

whereas when $z = 3$, $p_c^{(a)} = \tfrac{1}{2}$ and $P^{(a)}(p) = 0$ for all p.

Using equations (A6), (A7), (A8) and (A9) we get

$$S^{(a)}(p) = pR_1^z + (1-p)R_2^z - zpR_1^{z-1}\frac{\partial R_1}{\partial h}\bigg|_{h=0} - z(1-p)R_2^{z-1}\frac{\partial R_2}{\partial h}\bigg|_{h=0} \tag{A15}$$

$$\frac{\partial R_1}{\partial h}\bigg|_{h=0} = -\frac{(1-p)R_2^{z-1} + p(1-p)(z-1)R_1^{z-1}R_2^{z-2}}{1 - p(1-p)(z-1)^2(R_1R_2)^{z-2}} \tag{A16}$$

$$\frac{\partial R_2}{\partial h}\bigg|_{h=0} = -\frac{pR_1^{z-1} + p(1-p)(z-1)R_1^{z-2}R_2^{z-1}}{1 - p(1-p)(z-1)^2(R_1R_2)^{z-2}}. \tag{A17}$$

When $p \le p_c^{(a)}$, $R_1 = R_2 = 1$ and

$$S^{(a)}(p) = \frac{1 + (3z-1)[1/(z-1)^2 - \varepsilon(1-2p_c^{(a)}) - \varepsilon^2]}{\varepsilon(1-2p_c^{(a)})(z-1)^2 + \varepsilon^2(z-1)^2} \tag{A18}$$

so that when $z > 3$

$$S^{(a)}(p) = \frac{z(z+1)}{(z-1)^4(1-2p_c^{(a)})}\varepsilon^{-1} + O(1) \tag{A19}$$

and when $z = 3$

$$S^{(a)}(p) = \tfrac{3}{4}\varepsilon^{-2} + O(1). \tag{A20}$$

At the critical probability we may write

$$R_{1(2)}(p_c^{(a)}, h) = 1 - \mu_{1(2)} \tag{A21}$$

and using equations (A7), (A8) and (A9) we get

$$\mu_1 = \frac{1}{z-1}\left(\frac{2}{p_c^{(a)}(z-2)}\right)^{1/2} h^{1/2} + O(h)$$

$$\mu_2 = p_c^{(a)}(z-1)\mu_1 + O(h) \tag{A22}$$

and finally

$$P^{(a)}(p_c^{(a)}, h) = z\left(\frac{2p_c^{(a)}}{z-2}\right)^{1/2}\left(\frac{z}{z-1} - p_c^{(a)}\right)h^{1/2} + O(h). \tag{A23}$$

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